

Ginzburg - Landau theory of the triplet superconductivity in 3D Dirac semi-metal.

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It was recently shown that conventional phonon-electron interactions induce triplet pairing states in time-reversal invariant 3D Dirac semi - metals provided magnetic impurities or exchange interactions are strong enough¹. The order parameter in this case is a vector field. Starting from the microscopic model of the isotropic Dirac semi-metal, the Ginzburg-Landau energy for this field is derived using the Gor'kov technique. It was found that the transversal coherence length ξ_T is much smaller than the longitudinal, $\xi_L = 4\sqrt{2}\xi_T$, despite the isotropy. Several new features appear when an external field is applied. The Ginzburg - Landau model is used to investigate magnetic properties of such superconductors. Using the small deviation method the magnetic penetration depth was found also to be vastly different for longitudinal and transverse fluctuations $\lambda_T/\lambda_L = 4\sqrt{2}$. As a result the superconductor responds as type I to a transverse perturbation, while with respect to a longitudinal perturbation it behaves as type II. At large fields the order parameter orients itself perpendicular to the field direction. The triplet superconductor persists at arbitrarily high magnetic field (no upper critical magnetic field) like in some p wave superconductors.

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I. INTRODUCTION

Recently 3D Dirac semi - metals (DSM) like Na_3Bi and Cd_3As_2 with electronic states described by Bloch wave functions, obeying the "pseudo-relativistic" Dirac equation (with the Fermi velocity v_F replacing the velocity of light) were observed¹ and attracted widespread attention. The discovery of the 3D Dirac materials makes it possible to study their physics including remarkable electronic properties. This is rich in new phenomena like giant diamagnetism that diverges logarithmically when the chemical potential approaches the 3D Dirac point, a linear-in-frequency AC conductivity that has an imaginary part², quantum magnetoresistance showing linear field dependence in the bulk³. Most of the properties of these new materials were measured at relatively high temperatures. However recent experiments at low temperature on topological insulators and suspected 3D Dirac semi-metals exhibit superconductivity. Early attempts to either induce or discover superconductivity in Dirac materials were promising. The well known topological insulator Bi_2Se_3 doped with Cu , becomes superconducting at $T_c = 3.8K$ ⁴. At present its pairing symmetry is unknown. Some experimental evidence⁶ point to a conventional phononic pairing mechanism. The spin independent part of the effective electron - electron interaction due to phonons was studied theoretically⁷. For a conventional parabolic dispersion relation, typically independent of spin, the phonon mechanism leads to the s -wave superconductivity. The layered, non-centrosymmetric heavy element $PbTaSe_2$ was found to be superconducting⁴. Its electronic properties like specific heat, electrical resistivity, and magnetic-susceptibility indicate that $PbTaSe_2$ is a moderately coupled, type-II BCS superconductor with large electron-phonon coupling constant of $\lambda = 0.74$. It was shown theoretically to possess a very asymmetric 3D Dirac point created by strong spin-orbit coupling. If the 3D is con-

firmed, it might indicate that the superconductivity is conventional phonon mediated.

More recently when the Cu doped Bi_2Se_3 was subjected to pressure⁸, T_c increased to $7K$ at $30GPa$. Quasi-linear temperature dependence of the upper critical field H_{c2} , exceeding the orbital and Pauli limits for the singlet pairing, points to the triplet superconductivity. The band structure of the superconducting compounds is apparently not very different from its parent compound Bi_2Se_3 , so that one can keep the two band $\mathbf{k} \cdot \mathbf{p}$ description (Se p_z orbitals on the top and bottom layer of the unit cell mixed with its neighboring Bi p_z orbital). Electronic-structure calculations and experiments on the compounds under pressure⁸ reveal a single bulk three-dimensional Dirac cone like in Bi with large spin-orbit coupling. Moreover very recently some pnictides were identified as exhibiting Dirac spectrum⁹. This effort recently culminated in discovery of superconductivity in Cd_3As_2 ⁵. It is claimed that the superconductivity is p -wave at least on the surface.

The case of the Dirac semi-metals is very special due to the strong spin dependence of the itinerant electrons' effective Hamiltonian. It was pointed out¹⁰ that in this case the triplet possibility can arise although the triplet gap is smaller than that of the singlet, the difference sometimes is not large for spin independent electron - electron interactions. Very recently the spin dependent part of the phonon induced electron - electron interaction was considered¹¹ and it was shown that the singlet is still favored over the triplet pairing. Another essential spin dependent effective electron-electron interaction is the Stoner exchange among itinerant electrons leading to ferromagnetism in transition metals. While in the best 3D Weyl semi-metal candidates it is too small to form a ferromagnetic state, it might be important to determine the nature of the superconducting condensate. It turns out that it favors the triplet pairing¹². Also a modest concentration of magnetic impurities makes the triplet

ground state stable.

As mentioned above generally the applied magnetic field is an ultimate technique to probe the superconducting state. In a growing number of experiments, in addition to magnetotransport, magnetization curves, the magnetic penetration depth and upper critical magnetic field were measured¹³. It is therefore of importance to construct a Ginzburg - Landau (GL) description¹⁴ of these novel materials to study inhomogeneous order parameter configurations (junctions, boundaries, etc.) and magnetic properties that typically involve inhomogeneous configurations (like vortices) not amenable to a microscopic description.

In the present paper we derive such a GL type theory for triplet superconductor from the microscopic isotropic DSM model with attractive local interaction. The order parameter in this case is a vector field and the GL theory of vector field already considered in literature¹⁵⁻¹⁷ in connection with putative p - wave superconductors have several extraordinary features, both quantitative and qualitative.

The paper is organized as follows. The model of the (phonon mediated or unconventional) local interactions of 3D Dirac fermion is presented and the method of its solution (in the Gorkov equations form) including the symmetry analysis of possible pairing channels and the vectorial nature of the triplet order parameter is given in Section II. In Section III the Gorkov formalism, sufficiently general to derive the GL equations, is briefly presented. The most general form of the GL energy of the triplet superconductor in magnetic field consistent with the symmetries is given in IV. The coefficient of the relevant terms are calculated from the microscopic DSM model in section V. Section VI is devoted to applications of the GL model. The ground state degeneracy, the character of its excitations and basic magnetic properties are discussed. The vector order parameter is akin to optical phonons with sharp distinction between transverse and longitudinal modes. Transverse and longitudinal coherence lengths and penetration depths are calculated and the upper critical magnetic field is discussed. Section VI includes generalizations to include Pauli paramagnetism, discussion of an experimental possibility of observation of the excitation and conclusion.

II. THE LOCAL PAIRING MODEL IN THE DIRAC SEMI-METAL.

A. Pairing Hamiltonian in the Dirac semi-metal.

Electrons in the 3D Dirac semimetal are described by field operators $\psi_{fs}(\mathbf{r})$, where $f = L, R$ are the valley index (pseudospin) for the left/right chirality bands with spin projections taking the values $s = \uparrow, \downarrow$ with respect to, for example, the z axis. To use the Dirac ("pseudo-relativistic") notations, these are combined into a four component bi-spinor creation operator, $\psi^\dagger =$

$(\psi_{L\uparrow}^\dagger, \psi_{L\downarrow}^\dagger, \psi_{R\uparrow}^\dagger, \psi_{R\downarrow}^\dagger)$, whose index $\gamma = \{f, s\}$ takes four values. The non-interacting massless Hamiltonian with Fermi velocity v_F and chemical potential μ reads¹⁸

$$K = \int_{\mathbf{r}} \psi^\dagger(\mathbf{r}) \hat{K} \psi(\mathbf{r}); \quad (1)$$

$$\hat{K}_{\gamma\delta} = -i\hbar v_F \nabla^i \alpha_{\gamma\delta}^i - \mu \delta_{\gamma\delta}, \quad (2)$$

where the three 4×4 matrices, $i = x, y, z$,

$$\alpha = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad (3)$$

are presented in the block form via Pauli matrices σ . They are related to the Dirac γ matrices (in the chiral representation, sometimes termed "spinor") by $\alpha = \beta\gamma$ with

$$\beta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \quad (4)$$

Here $\mathbf{1}$ is the 2×2 identity matrix.

We consider a special case of 3D rotational symmetry that in particular has an isotropic Fermi velocity. Moreover we assume time reversal, $\Theta\psi(\mathbf{r}) = i\sigma_y\psi^*(\mathbf{r})$, and inversion symmetries although the pseudo-Lorentz symmetry will be explicitly broken by interactions. The spectrum of single particle excitations is linear. The chemical potential μ is counted from the Dirac point.

As usual in certain cases the actual interaction can be approximated by a model local one:

$$V_{eff} = -\frac{g}{2} \int_{\mathbf{r}} \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r}) \psi_\alpha(\mathbf{r}). \quad (5)$$

Unlike the free Hamiltonian K , Eq.(1), this interaction Hamiltonian does not mix different spin components.

Spin density in Dirac semi-metal has the form

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2} \psi^\dagger(\mathbf{r}) \boldsymbol{\Sigma} \psi(\mathbf{r}), \quad (6)$$

where the matrices

$$\boldsymbol{\Sigma} = -\alpha\gamma_5 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad (7)$$

$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad (8)$$

are also the rotation generators.

B. The symmetry classification of possible pairing channels.

Since we consider the local interactions as dominant, the superconducting condensate (the off-diagonal order parameter) will be local

$$O = \int_{\mathbf{r}} \psi_\alpha^\dagger(\mathbf{r}) M_{\alpha\beta} \psi_\beta^\dagger(\mathbf{r}), \quad (9)$$

where the constant matrix M should be a 4×4 anti-symmetric matrix. Due to the rotation symmetry they transform covariantly under infinitesimal rotations generated by the spin S^i operator, Eq.(6):

$$\int_{\mathbf{r}, \mathbf{r}'} \left[\psi_{\alpha}^{+}(r) M_{\alpha\beta} \psi_{\beta}^{+}(r), \psi_{\gamma}^{+}(r') \Sigma_{\gamma\delta}^i \psi_{\delta}^{+}(r') \right] \quad (10)$$

$$= - \int_r \psi_{\gamma}^{+}(r) (\Sigma_{\gamma\delta}^i M_{\delta\kappa} + M_{\gamma\delta} \Sigma_{\delta\gamma}^{ti}) \psi_{\kappa}^{+}(r). \quad (11)$$

Here and in what follows "t" denotes the transpose matrix. The representations of the rotation group therefore characterize various possible superconducting phases.

Out of 16 matrices of the four dimensional Clifford algebra six are antisymmetric and one finds one vector and three scalar multiplets of the rotation group. The multiplets contain:

(i) a triplet of order parameters:

$$\{M_x^T, M_y^T, M_z^T\} \quad (12)$$

$$= \{-\beta\alpha_z, -i\beta\gamma_5, \beta\alpha_x\} = \{T_x, T_y, T_z\} \quad (13)$$

The algebra is

$$\Sigma_i T_j + T_j \Sigma_i^t = 2i\varepsilon_{ijk} T_k. \quad (14)$$

Note that the three matrices T_i are Hermitian.

(ii) three singlets

$$M_1^S = i\alpha_y; \quad M_2^S = i\Sigma_y; \quad M_3^S = -i\beta\alpha_y\gamma_5. \quad (15)$$

Which one of the condensates is realized at zero temperature is determined by the parameters of the Hamiltonian and is addressed next within the Gaussian approximation. As was shown in our previous work¹² either exchange interactions or magnetic impurities make the triplet state a leading superconducting channel in these materials. Therefore we will consider in the next section only the vector channel.

III. GORKOV EQUATIONS AND THE TRIPLET PAIRING

A. Gorkov equations for Green's functions in matrix form

Using the standard BCS formalism, the Matsubara Green's functions (τ is the Matsubara time)

$$G_{\alpha\beta}(\mathbf{r}, \tau; \mathbf{r}', \tau') = - \langle T_{\tau} \psi_{\alpha}(\mathbf{r}, \tau) \psi_{\beta}^{\dagger}(\mathbf{r}', \tau') \rangle; \quad (16)$$

$$F_{\alpha\beta}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle T_{\tau} \psi_{\alpha}(\mathbf{r}, \tau) \psi_{\beta}(\mathbf{r}', \tau') \rangle;$$

$$F_{\alpha\beta}^{+}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle T_{\tau} \psi_{\alpha}^{\dagger}(\mathbf{r}, \tau) \psi_{\beta}^{\dagger}(\mathbf{r}', \tau') \rangle,$$

obey the Gor'kov equations¹⁹:

$$\begin{aligned} & -\frac{\partial G_{\gamma\kappa}(\mathbf{r}, \tau; \mathbf{r}', \tau')}{\partial \tau} - \int_{\mathbf{r}''} \langle \mathbf{r} | \hat{K}_{\gamma\beta} | \mathbf{r}'' \rangle G_{\beta\kappa}(\mathbf{r}'', \tau; \mathbf{r}', \tau') \\ & - g F_{\beta\gamma}(\mathbf{r}, \tau; \mathbf{r}, \tau) F_{\beta\kappa}^{+}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \delta^{\gamma\kappa} \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - \tau'); \\ & \frac{\partial F_{\gamma\kappa}^{+}(\mathbf{r}, \tau; \mathbf{r}', \tau')}{\partial \tau} - \int_{\mathbf{r}''} \langle \mathbf{r} | \hat{K}_{\gamma\beta}^t | \mathbf{r}'' \rangle F_{\beta\kappa}^{+}(\mathbf{r}'', \tau; \mathbf{r}', \tau') \\ & - g F_{\gamma\beta}^{+}(\mathbf{r}, \tau; \mathbf{r}, \tau) G_{\beta\kappa}(\mathbf{r}, \tau; \mathbf{r}', \tau') = 0. \end{aligned} \quad (17)$$

These equations are conveniently presented in matrix form (superscript t denotes transposed and I - the identity matrix):

$$\begin{aligned} & \int_{X''} D^{-1}(X, X'') G(X'', X') - \Delta(X) F^{+}(X, X') = I \delta(X - X'); \\ & \int_{X''} D^{t-1}(X, X'') F^{+}(X'', X') + \Delta^{t*}(X) G(X, X') = 0. \end{aligned} \quad (18)$$

Here $X = (\mathbf{r}, \tau)$, $\Delta_{\alpha\beta}(X) = g F_{\beta\alpha}(X, X)$ and

$$D_{\alpha\beta}^{-1}(X, X') = -\delta_{\alpha\beta} \frac{\partial}{\partial \tau} \delta(X - X') - \delta(\tau - \tau') \langle \mathbf{r} | \hat{K}_{\alpha\beta} | \mathbf{r}' \rangle. \quad (19)$$

In the homogeneous case the Gor'kov equations for Fourier components of the Green's functions simplify considerably:

$$\begin{aligned} & D^{-1}(\omega, \mathbf{p}) G(\omega, \mathbf{p}) - \Delta F^{+}(\omega, \mathbf{p}) = I; \\ & \tilde{D}^{-1}(\omega, \mathbf{p}) F^{+}(\omega, \mathbf{p}) + \Delta^{t*} G(\omega, \mathbf{p}) = 0. \end{aligned} \quad (20)$$

The matrix gap function can be chosen as

$$\Delta_{\beta\gamma} = g F_{\gamma\beta}(0) = \Delta_z M_{\gamma\beta}, \quad (21)$$

with real constant Δ_z . Here $D^{-1}(\omega, \mathbf{p}) = i\omega + \mu - \boldsymbol{\alpha} \cdot \mathbf{p}$, is the noninteracting inverse Dirac Green's function for the Hamiltonian Eq.(1) and $\tilde{D}^{-1}(\omega, \mathbf{p}) = i\omega - \mu - \boldsymbol{\alpha}^t \cdot \mathbf{p}$, where $\omega_n = \pi T(2n + 1)$ is the fermionic Matsubara frequency.

Solving these equations one obtains (in matrix form)

$$\begin{aligned} & G^{-1} = D^{-1} + \Delta \tilde{D} \Delta^{t*}; \\ & F^{+} = -\tilde{D} \Delta^{t*} G, \end{aligned} \quad (22)$$

with the gap function to be found from the consistency condition

$$\Delta^{t*} = -g \sum_{\omega_p} \tilde{D} \Delta^{t*} G. \quad (23)$$

Now we find solutions of this equation for each of the possible superconducting phases.

B. Homogeneous triplet solution of the gap equation.

In this phase rotational symmetry is spontaneously broken simultaneously with the electric charge $U(1)$ (global gauge invariance) symmetry. Assuming z direction of the p -wave condensate the order parameter matrix takes a form:

$$\Delta = \Delta_z T_z = \Delta_z \beta \alpha_x, \quad (24)$$

where Δ_z is a constant. The energy scale will be set by the Debye cutoff T_D of the electron-phonon interactions, see below.

The spectrum of elementary excitations at zero temperature was discussed in ref.¹². There is a saddle point with energy gap $2\Delta_z$ on the circle $p_\perp^2 \equiv p_x^2 + p_y^2 = \mu^2/v_F^2$, $p_z = 0$. The gap Δ_z as a function of the dimensionless phonon-electron coupling $\lambda = gN$, where N being the density of states (all spins and valleys included), increases upon reduction in μ . At large $\mu \gg T_D$, as in BCS, the gap becomes independent of μ and one has the relation

$$\frac{1}{g} = \frac{N}{12} \sinh^{-1} \frac{T_D}{\Delta_z}; N = \frac{2\mu^2}{\pi^2 v_F^3 \hbar^3}, \quad (25)$$

leading to an exponential gap dependence on λ when it is small: $\Delta_z = T_D / \sinh(12/\lambda) \simeq 2T_D e^{-12/\lambda}$.

The critical temperature is obtained from Eq.(23) with discretized ω by substituting $\Delta_z = 0$. To utilize the orthonormality of T_i , $\text{Tr}(T_i T_j^*) = 4\delta_{ij}$, one multiplies the gap equation by the matrix T_z/g and takes the trace:

$$\frac{1}{g} \text{Tr}(T_z T_z^*) = \frac{4}{g} = T_c \mathcal{B}_{zz}. \quad (26)$$

The bubble integral is

$$\mathcal{B}_{ij} = \sum_{\mathbf{p}} \text{Tr} \left(T_i \tilde{D} T_j^* D \right) = 4\delta_{ij} T_c \times \quad (27)$$

$$\times \sum_{n\mathbf{p}} \frac{v_F^2 (p_\perp^2 - p_z^2) + \mu^2 + \omega_n^2}{\omega_n^4 + (v_F^2 p^2 - \mu^2)^2 + 2\omega_n^2 (v_F^2 p^2 + \mu^2)} \quad (28)$$

Performing first the sum over Matsubara frequencies and then integrate over q one obtains, similarly to the singlet BCS, (see Appendix A for details):

$$T_c = \frac{2\gamma_E}{\pi} T_D e^{-12/\lambda}, \quad (29)$$

where $\log \gamma_E = 0.577$ is the Euler constant.

IV. A GENERAL GL DESCRIPTION OF A TRIPLET SUPERCONDUCTOR IN A MAGNETIC FIELD.

In this section the effective description of the superconducting condensate in terms of the varying (on the mesoscopic scale) order complex parameter vector field $\Delta_i(\mathbf{r})$ is presented.

A. The GL description for a vector order parameter

The static phenomenological description is determined by the GL free energy functional $F[\Delta(\mathbf{r}), \mathbf{A}(\mathbf{r})]$ expanded to second order in gradients and fourth order in Δ . In a magnetic field \mathbf{B} , as usual, space derivatives of the microscopic Hamiltonian become covariant derivatives $\nabla \rightarrow \mathcal{D} = \nabla + i \frac{e^*}{\hbar c} \mathbf{A}$, $e^* = 2e$ due to gauge invariance under $\Delta_i \rightarrow e^{i\chi(\mathbf{r})} \Delta_i$, $A_i \rightarrow A_i - \frac{\hbar c}{e^*} \nabla \chi$. Naively the only modification of the GL energy is in the gradient term, Eq.(30); the most general gradient term consistent with rotation symmetry and the $U(1)$ gauge symmetry is

$$F_{\text{grad}} = N \int_{\mathbf{r}} \left\{ u_T \{ (\mathcal{D}_j \Delta_i)^* (\mathcal{D}_j \Delta_i) - (\mathcal{D}_i \Delta_j)^* (\mathcal{D}_j \Delta_i) \} + u_L (\mathcal{D}_i \Delta_j)^* (\mathcal{D}_j \Delta_i) \right\}. \quad (30)$$

The factor N , the density of states, is customarily introduced into energy¹². It was noted in¹⁶, that, unlike in the usual scalar order parameter case, the longitudinal and transverse coefficients are in general different, leading to two distinct coherence lengths, see Section V. Possibilities for the local terms are¹⁵

$$F_{\text{loc}} = N(\mu) \int_{\mathbf{r}} \left\{ \alpha (T - T_c) \Delta_i^* \Delta_i + \frac{\beta_1}{2} (\Delta_i^* \Delta_i)^2 + \frac{\beta_2}{2} |\Delta_i \Delta_i|^2 \right\}. \quad (31)$$

The magnetic part, $F_{\text{mag}} = B^2/8\pi$, completes the free energy.

B. The set of GL equations for triplet order parameter

The set of the GL equations corresponding to this energy are obtained by variation with respect to Ψ_j^* and A_i . The first is:

$$- \left\{ u_T \left(\delta_{ij} \mathcal{D}^2 - \frac{1}{2} \{ \mathcal{D}_i, \mathcal{D}_j \} \right) + \frac{1}{2} u_L \{ \mathcal{D}_i, \mathcal{D}_j \} \right\} \Delta_j + \quad (32)$$

$$+ \alpha (T - T_c) \Delta_i + \beta_1 \Delta_i \Delta_j^* \Delta_j + \beta_2 \Delta_i^* \Delta_j \Delta_j = 0.$$

The anticommutator appears due complex conjugate terms in Eq.(30)²⁰. The Maxwell equation for the supercurrent density is:

$$J_i = \frac{ie^*}{\hbar} N(\mu) \left\{ u_T \Delta_j^* \mathcal{D}_i \Delta_j + u_L \Delta_j^* \mathcal{D}_j \Delta_i \right\} + cc, \quad (33)$$

where $u = u_L - u_T$.

Having established coefficients $u_{T,L}, \beta_{1,2}$, μ_Z and α , our aim in the next Section is to deduce them from the microscopic Dirac semi-metal model.

V. GL COEFFICIENTS FROM THE GOR'KOV EQUATIONS

For the calculation of coefficients of the local part, the homogeneous Gor'kov equation, Eq.(23) suffices, while for calculation of the gradient terms a general linearized equation, Eq.(18) is necessary. Magnetic field effects are required only for the calculation of the Zeeman term coefficients and normalization of the order parameter via supercurrent density.

A. Local (potential) terms in Gor'kov

Iterating once the equation Eq.(23) with help of Eq.(22) one obtains the local terms to third order in the gap function:

$$\frac{1}{g}\Delta^{*t} + \sum_{\omega \mathbf{p}} \left\{ \begin{array}{l} \tilde{D}(\omega, \mathbf{p}) \Delta^{*t} D(\omega, \mathbf{p}) - \\ - \tilde{D}(\omega, \mathbf{p}) \Delta^{*t} D(\omega, \mathbf{p}) \Delta \tilde{D}(\omega, \mathbf{p}) \Delta^{*t} D(\omega, \mathbf{p}) \end{array} \right\} \quad (34)$$

Using $\Delta^{t*} = \Delta_i^* T_i$, multiplying by T_i^t and taking the trace, one gets the linear local terms

$$N \alpha (T - T_c) \Delta_i^* = \frac{4}{g} \Delta_i^*, \quad (35)$$

where the bubble integral was given in Eq.(27). Expressing g via T_c , see Eq.(29) allows to write the coefficient α of Δ_i^* in the Gorkov equation Eq.(23) is

$$\alpha (T - T_c) = \frac{8\mu^2}{3\pi^2 v_F^3 \hbar^3 N} \log \frac{T}{T_c} \approx \frac{4}{3} \frac{T - T_c}{T_c}, \quad (36)$$

where N is the density of states.

The cubic terms in Eq.(34), multiplied again by T_i^t and "traced" take the form

$$\begin{aligned} & N (\beta_1 \Delta_j^* \Delta_j \Delta_i^* + \beta_2 \Delta_j^* \Delta_j^* \Delta_i) \\ & = -\Delta_j^* \Delta_k \Delta_l^* \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_i^t \tilde{D} T_j^{t*} D T_k \tilde{D} T_l^{t*} D \right\}. \end{aligned} \quad (37)$$

$$= -\Delta_j^* \Delta_k \Delta_l^* \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_i^t \tilde{D} T_j^{t*} D T_k \tilde{D} T_l^{t*} D \right\}. \quad (38)$$

The calculation is given in Appendix A and results in:

$$\beta_1 = \frac{7\zeta(3)}{20\pi^2} \frac{1}{T_c^2}; \quad \beta_2 = -\frac{1}{3}\beta_1. \quad (39)$$

The Riemann zeta function is $\zeta(3) = 1.2$.

B. Linear gradient terms

To calculate the gradient terms, one first linearizes the Gor'kov equations, Eq.(18)

$$\int_{X''} D^{-1}(X, X'') G(X'', X') \quad (40)$$

$$= I \delta(X - X') \rightarrow G = D^{-1}; \quad (41)$$

$$\begin{aligned} & F^+(X, X') \\ & = - \int_{X''} D^t(X - X'') \Delta^{*t}(X'') D(X'' - X'). \end{aligned} \quad (42)$$

In particular,

$$\frac{1}{g} \Delta^{*t}(X) = F^+(X, X) \quad (43)$$

$$= - \int_{X'} D^t(X - X') \Delta^{*t}(X') D(X' - X) \quad (44)$$

The anomalous Green's functions are no longer space translation invariant, so that the following Fourier transform is required: The (time independent) order parameter is also represented via Fourier components $\Delta^*(X) = \sum_{\mathbf{P}} e^{-i\mathbf{P}\cdot\mathbf{r}} \Delta^*(\mathbf{P})$. The linear part Gor'kov equation (this time including nonlocal parts) as the only "external" momentum \mathbf{P} reads

$$\frac{1}{g} \Delta^{*t}(\mathbf{P}) + \sum_{\omega \mathbf{p}} \tilde{D}(\omega, \mathbf{p}) \Delta^{t*}(\mathbf{P}) D(\omega, \mathbf{p} - \mathbf{P}). \quad (45)$$

To find the coefficients of the gradient terms, one should consider contributions quadratic in P from the expansion of both $\Delta^{t*}(\mathbf{P})$ and $D(\omega, \mathbf{p} - \mathbf{P})$. In view of the gap equation Eq.(26,27), the expansions of $\Delta^{t*}(\mathbf{P})$ cancel each other up to small corrections of order $T - T_c$. So that multiplying by T_i^t and taking the trace

$$\frac{1}{2} P_k P_l \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_i^t \tilde{D}(\omega, \mathbf{p}) T_j^{t*} D_{kl}''(\omega, \mathbf{p}) \right\} \Delta_j^*, \quad (46)$$

where $D_{kl}''(\omega, \mathbf{p}) \equiv \frac{\partial^2 D(\omega, \mathbf{p})}{\partial p_k \partial p_l}$. Comparing this with the gradient terms in the GL equation, Eq.(32), see Appendix B for details, one deduces

$$u_T = \frac{28\zeta(3)}{15\pi^2} \frac{v_F^2 \hbar^2}{T_c^2}; \quad u_L = \frac{1}{32} u_T. \quad (47)$$

Note the very small longitudinal coefficient, $u_L \ll u_T$. As we shall see in the following section it has profound phenomenological consequences.

VI. BASIC PROPERTIES OF THE TRIPLET SUPERCONDUCTOR

A. Ground state structure and degeneracy

A ground state is characterized by three independent parameters corresponding to three Goldstone bosons. The GL energy is invariant under both the vector $SO(3)$ rotations, $\Delta_i \rightarrow R_{ij} \Delta_j$, and the superconducting phase $U(1)$, $\Delta_i \rightarrow e^{i\chi} \Delta_i$. In the superconducting state characterized by the vector order parameter Δ ($|\Delta| = \Delta$, energy gap) the $U(1)$ is broken: $U(1) \rightarrow 1$, while the $SO(3)$ is only partially broken down to its $SO(2)$. There are therefore three Goldstone modes. Here we explicitly parametrize these degrees of freedom by phases following ref.¹⁵. Generally a complex vector field can be written as

$$\Delta = \Delta (\mathbf{n} \cos \phi + i \mathbf{m} \sin \phi), \quad (48)$$

where \mathbf{n} and \mathbf{m} are arbitrary unit vectors and $0 < \phi < \pi/2$. Using this parametrization the homogeneous part of the free-energy density, Eq.(31), takes the form

$$f_{loc} = N \left\{ \frac{\alpha (T - T_c) \Delta^2}{+ \frac{1}{2} (\beta_1 + \beta_2 (\cos^2(2\phi) + (\mathbf{n} \cdot \mathbf{m})^2 \sin^2(2\phi)))} \right\} \Delta^4 \quad (49)$$

This form allows us to make several interesting observations. The crucial sign is that of β_2 . In previous studies^{15,21} only $\beta_2 > 0$ (so called phase A) was considered. In our case however $\beta_2 < 0$ and different ground state configurations should be considered. In phase B the minimization gives, $\mathbf{n} = \pm \mathbf{m}$. Note two different solutions. So that the "vacuum manifold" is

$$\Delta = \Delta_0 \mathbf{n} e^{i\chi}. \quad (50)$$

Here the range of χ was enlarged, $-\pi/2 < \chi < \pi/2$, to incorporate $\mathbf{n} = \pm \mathbf{m}$. One can use the spherical representation of the unit vector (that will be termed "director"): $\mathbf{n} = (\sin \theta_n \cos \varphi_n, \sin \theta_n \sin \varphi_n, \cos \theta_n)$. The ground state energy density therefore is achieved at

$$\Delta_0^2 = \frac{\alpha (T_c - T)}{\beta_1 + \beta_2}. \quad (51)$$

Mathematically the vacuum manifold in phase B is isomorphic to $S_2 \otimes S_1/Z_2$. This determines the thermodynamics of the superconductor very much in analogy with the scalar superconductor with $\beta = \beta_1 + \beta_2$. Magnetic properties are however markedly different.

B. Small fluctuations analysis: two vastly different penetration depths and coherence lengths.

Here the response of the superconductor in phase B to an external perturbation, like boundary or magnetic field, is considered. The basic excitation modes are uncovered by the linear stability analysis very similar to the so-called Anderson - Higgs mechanism in field theory applied to (scalar order parameter) superconductivity a long time ago²³. Two basic scales, the coherence length (scale of variations of the order parameter) and magnetic penetration depth (scale of variations of the magnetic field), are obtained from the expansion of the GL energy to second order in fluctuations around superconducting ground state at zero field. The fluctuations are parametrized by $\theta_n, \varphi_n, \theta_m, \varphi_m, \phi, A_i$, $\Delta = \Delta_0 (1 + \varepsilon)$. The order parameter, Eq.(48), to the second order is

$$\Delta/\Delta_0 \approx (0, 0, 1) + (\theta_n, 0, \varepsilon + i\phi) \quad (52)$$

$$+ (\varepsilon \theta_n + i \theta_m \phi, \theta_n \varphi_n - \theta_n^2/2 + i \varepsilon \phi - \phi^2/2) \quad (53)$$

For a perturbation with wave vector \mathbf{k} the energy to quadratic order in fluctuations reads:

$$\frac{\delta F}{N} = \sum_{\mathbf{k}} \left\{ \frac{\alpha(T_c - T)}{\beta_1 + \beta_2} \{ 2\alpha (T_c - T) \varepsilon^* \varepsilon + u_T W_T + u W \} + \frac{\hbar^2 c^2}{8\pi e^* N} A_i^* (k^2 \delta_{ij} - k_i k_j) A_j \right\}, \quad (54)$$

where

$$W_T = k^2 (\theta_n^* \theta_n + \varepsilon^* \varepsilon + \phi^* \phi + i (\varepsilon^* \phi - \phi^* \varepsilon)) \quad (55)$$

$$+ (\varepsilon^* - i \phi^*) k_j A_j + k_j A_j^* (\varepsilon + i \phi) + A_j^* A_j; \quad (56)$$

$$W = k_1^2 \theta_n^* \theta_n + k_1 k_3 \theta_n^* (\varepsilon + i \phi) + k_1 k_3 (\varepsilon^* - i \phi^*) \theta_n + k_3^2 (\varepsilon^* \varepsilon + \phi^* \phi + i (\varepsilon^* \phi - \phi^* \varepsilon)) \quad (57)$$

$$+ k_3 \left(\theta_n^* A_1 + (\varepsilon^* - i \phi^*) A_3 + A_1^* \theta_n \right) + A_3^* A_3.$$

Diagonalization of this quadratic form reveals three Goldstone modes (that do not affect the characteristic lengths) and the "massive" fields, ε and \mathbf{A} (the only one contributing for $\mathbf{k} = 0$) with different longitudinal and transversal characteristic lengths:

$$\xi_T^2 = \frac{u_T}{2\alpha (T_c - T)}; \quad \xi_L^2 = \frac{u_L}{u_T} \xi_T^2; \quad (58)$$

$$\lambda_T^2 = \frac{\hbar^2 c^2 (\beta_1 + \beta_2)}{8\pi e^* u_T \alpha (T_c - T) N}; \quad \lambda_L^2 = \frac{u_T}{u_L} \lambda_T^2.$$

Here ξ_T is the coherence length along the directions perpendicular to the vector \mathbf{n} , while the one parallel to \mathbf{n} is ξ_L .

Similarly for magnetic penetration depths $\lambda_{T,L}$. Our calculation in the previous Section for the Dirac semi-metal, see Eq.(47), demonstrate that both are quite different since $u_L/u_T = 1/32 \ll 1$. This is obviously of great importance for large magnetic field properties of such superconductors. The phenomenological consequences for vortex state are briefly discussed in Section VI.

C. Strong magnetic fields: is there an upper critical field H_{c2} ?

In strong homogeneous magnetic field H (assumed to be directed along the z axis) superconductivity typically (but not always, see an example of the p -wave superconductor that develops flux phases²²) disappears at certain critical value H_{c2} . This bifurcation point is determined within the GL framework by the lowest eigenvalue of the linearized GL equations. This is an exact requirement of stability of the normal phase^{14,24}. The linearized GL equation Eq.(32) reads:

$$\left[(a - u_T D^2) \delta_{ij} - \frac{u}{2} \{ \mathcal{D}_i, \mathcal{D}_j \} \right] \Delta_j = 0, \quad (59)$$

where coefficients are in Eq.(47), and $u = u_L - u_T$. We use the Landau gauge, $A_x = H_{c2} y$; $A_y = A_z = 0$. Assuming translation symmetry along the field direction,

$\partial_z \Delta_i = 0$, the operators of the eigenvalue problem depend on x and y only.

Since we have three components of the order parameter, there are three eigenvalues. It is easily seen from Eq.(59) that the z -component of the order parameter Δ_z parallel to the external field direction is independent of the other two, Δ_x, Δ_y , leading to the ordinary Abrikosov value:

$$-u_T \mathcal{D}^2 \Delta_z = -a \Delta_z \rightarrow H_{c2}^{\parallel} = \frac{\Phi_0}{2\pi \xi_T^2}, \quad (60)$$

where ξ_T is defined in Eq.(58). To avoid confusion with customary notations for layered materials (like high T_c cuprates), the material that is modelled here is isotropic and "parallel", "perpendicular" and refer to the relative orientation of the magnetic field to the vector order parameter rather than to a layer. The orientation of the order parameter in isotropic material considered here, due to degeneracy of the ground state, is determined by the external magnetic field as we exemplify next.

The two remaining eigenvalues involving only the order parameter components Δ_x and Δ_y perpendicular to the field are obtained from diagonalizing the "Hamiltonian":

$$\mathcal{H} \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} = -a \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix}; \quad (61)$$

$$\mathcal{H} = - \begin{pmatrix} u_T \mathcal{D}_y^2 + u_L \mathcal{D}_x^2 & \frac{u}{2} \{\mathcal{D}_x, \mathcal{D}_y\} \\ \frac{u}{2} \{\mathcal{D}_x, \mathcal{D}_y\} & u_T \mathcal{D}_x^2 + u_L \mathcal{D}_y^2 \end{pmatrix}.$$

This nontrivial eigenvalue problem fortunately can be solved exactly, see Appendix C. The lowest eigenstate being a superposition of just two lowest even Landau levels, $|0\rangle$ and $|2\rangle$ are: $\Delta_x = \alpha |0\rangle + \beta |2\rangle$, $\Delta_y = \gamma |0\rangle + i\beta |2\rangle$. The lowest of these eigenvalues is

$$\frac{e^* H_{c2}^{\perp}}{\hbar c} \left(-\sqrt{3 \left(\frac{3}{2} (u_T + u_L) \right) - 2u_T u_L} \right) = \alpha (T_c - T). \quad (62)$$

The corresponding critical field H_{c2}^{\perp} ("perpendicular" refers to the order parameter direction) that can be expressed via an effective "perpendicular" coherence length,

$$H_{c2}^{\perp} = \frac{\Phi_0}{2\pi \xi_{\perp}^2}; \quad (63)$$

$$\xi_{\perp}^2 = \frac{3}{2} (\xi_L^2 + \xi_T^2) - \sqrt{3\xi_L^4 + 3\xi_T^4 - 2\xi_L^2 \xi_T^2}. \quad (64)$$

It is always larger than H_{c2}^{\parallel} , and therefore is physically realized. The upper field H_{c2}^{\perp} becomes infinite at $r_c = u_L/u_T = (13 - 4\sqrt{10})/3 \simeq 0.117$. This means that in such material superconductivity persists at any magnetic field like in some p -wave superconductors. It was found in Section IV that for the simplest Dirac semi-metal, $r = 1/32 < r_c$ see Eq.(47). Thus there is no upper critical field in this case. Of course, different microscopic models that belong to the same universality class, might have

higher r . In any case the Abrikosov lattice is expected to be markedly different from the conventional one and even from the vector order parameter model studied in¹⁵. At large fields in principle there is a direct interaction between the external magnetic field and spin, not taken into account in Hamiltonian Eq.(1) studied so far. The next subsection addresses this question.

D. What impact has the Pauli paramagnetism at strong fields?

Since the prediction of the FFLO effect¹⁴ in low T_c superconductors it is well known that at very high magnetic fields the direct spin - magnetic field coupling on the microscopic level might not be negligible. The singlet channel Cooper pair is effectively "broken" by the splitting since the spins of the two electrons are opposite (Pauli paramagnetic limit). It is not clear what impact it has on Dirac semi-metals. If the impact is large it could be incorporated as an additional paramagnetic term in the GL energy. In an isotropic Dirac superconductor one has only one possible term in the GL energy term linear in paramagnetic coupling and consistent with symmetries:

$$F_{par} = N\mu_p \int_{\mathbf{r}} i (\boldsymbol{\Delta}^* \times \boldsymbol{\Delta}) \cdot \mathbf{B}, \quad (65)$$

where μ_p is the effective "spin" of the Cooper pair sometimes called "Zeeman coupling"^{15,24}.

The single particle Hamiltonian in magnetic field with the Pauli term becomes

$$\hat{K} = -iv_F \hbar \mathcal{D} \cdot \boldsymbol{\alpha} - \mu + \mu_B \boldsymbol{\Sigma} \cdot \mathbf{B}, \quad (66)$$

where the Bohr magneton, $\mu_B = e\hbar/2mc$, determines the strength of the coupling of the spin to magnetic field, with m being the free electron mass. The direct calculation, see Appendix A, shows that $\mu_p = 0$.

VII. DISCUSSION AND CONCLUSIONS

Starting from the microscopic model of the isotropic Dirac semi-metal, the Ginzburg-Landau energy for this field is derived using the Gor'kov technique. It was found that the transversal coherence length ξ_T is much smaller than the longitudinal, $\xi_L = 4\sqrt{2}\xi_T$, despite the isotropy. Several new features appear when an external field is applied. The Ginzburg - Landau model is used to investigate magnetic properties of such superconductors. Using the small deviation method the magnetic penetration depth was found also to be vastly different for longitudinal and transverse fluctuations $\lambda_T/\lambda_L = 4\sqrt{2}$.

We have observed that properties of the triplet superconductor phase of the Dirac semi-metal has extremely unusual features that we would like to associate qualitatively with the characteristics of the Cooper pair. The

superconducting state generally is a Bose - Einstein condensate of composite bosons - Cooper pairs, classically described by the Ginzburg - Landau energy as a functional of the order parameter. In the present case the Cooper boson is described by a *vector field* $\Delta_i(\mathbf{r})$. In this respect it is reminiscent to phonon and vector mesons in particle physics²³. Vector fields generally have both the orbital and internal degrees of freedom often called polarization. The internal degree of freedom might be connected to the "valley" degree of freedom of constituents of the composite boson. We have provided evidence that the Cooper pair in DSM has finite orbital momentum, albeit, as will be shown shortly, the spin magnetic moment is zero. Microscopically the unusual nature is related to the presence of the valley degeneracy in Dirac semi-metal. While in a single band superconductor the Pauli principle requires a triplet Cooper pair to have both odd angular momentum and spin, it is no longer the case in the Dirac semi-metal.

A massive bosonic vector field in isotropic situation (the case considered here) generally have distinct transversal and longitudinal polarizations (massless fields like photons in dielectric do not possess the longitudinal degree of freedom). The results for coherence lengths $\xi_{T,L}$ and the penetration depths $\lambda_{T,L}$ in triplet superconductor in DSM demonstrate pronounced disparity between properties of transverse and longitudinal polarizations. In conventional "scalar" order parameter superconductor the Abrikosov ratio¹⁴, $\kappa \equiv \lambda/\xi$, distinguishes between type I ($\kappa < \kappa_c = 1/\sqrt{2}$) and type II, where, for example, vortices appear under a magnetic field. In the present case this separation is ambiguous. There are two quite different ratios: transversal and the longitudinal, (in respect to director of the order parameter $n = \Delta/|\Delta|$). see Eq.(47).

$$\kappa_T = \frac{\lambda_T}{\xi_T} = \quad (67)$$

$$\frac{\hbar c}{e^* u_T} \sqrt{\frac{\beta_1 + \beta_2}{4\pi N}} = \frac{1}{16} \sqrt{\frac{15\pi^3}{7\zeta(3)}} \frac{c T_c \hbar^{1/2}}{e^* \mu v_F^{1/2}}; \quad (68)$$

$$\kappa_L = \frac{\lambda_L}{\xi_L} = \frac{u_T}{u_L} \kappa_T.$$

For typical DSM one estimates the Fermi velocity and chemical potential¹ $v_F = c/200$, $\mu = 0.2eV$, and with the expected critical temperature^{8,11} $T_c = 5K$, one obtains the transversal Abrikosov ratio $\kappa_T = 0.08$, smaller than critical, while the longitudinal, $\kappa_L = 2.7$ is larger. This means that longitudinal fluctuations the material behaves as type II, while response to the transverse ones is that of a type I superconductor. This has an impact on transport, optical and magnetic properties of these superconductors.

The vortex physics of strongly type II triplet superconductors of this type is very rich and some of it has already been investigated in connection with heavy fermion and other superconductors suspected to possess *p*-wave pairing. In particular, their magnetic vortices appear as

either vector vortices or so-called skyrmions¹⁵ - coreless topologically nontrivial textures. The magnetic properties like the magnetization are very peculiar and even without a magnetic field the system forms a "spontaneous flux state". The material therefore can be called a "ferromagnetic superconductor". The superconducting state develops weak ferromagnetism and a system of alternating magnetic domains²¹. It was noted¹⁵ that the phase is reminiscent to the phase B of superfluid He_3 .¹⁴ (with an obvious distinction that the order parameter in the later case is neutral rather than charged and tensorial rather than vectorial). As was shown in section V, the Dirac semi-metal triplet superconductor phase is different in several respects. It is more like the phase B of superfluid He_3 .

Experimentally the major consequence of the present theoretical investigation, namely the polarization effect of the vector order parameter should be pronounced in the AC response of these materials. Recently the AC response of the disordered superconductor was utilized to probe Goldstone modes²⁵. We have demonstrated that they are abundant in the triplet DSM superconductor.

VIII. APPENDIX A.

A. Critical temperature calculation

Starting from equation Eq.(26) the angle integrations result in (for $\mu \gg T_D, T_c$)

$$\begin{aligned} \frac{1}{g} &= T \sum_{np} \frac{\mu^2 + \omega_n^2}{\omega_n^4 + (p^2 - \mu^2)^2 + 2\omega_n^2(p^2 + \mu^2)} \quad (69) \\ &= \frac{\mu^2}{12\pi^2} \int_{\varepsilon=-T_D}^{T_D} \frac{\tanh(\varepsilon/2T)}{\varepsilon} \approx \frac{\mu^2}{6\pi^2} \log \frac{2T_D \gamma_E}{\pi T_c}. \end{aligned}$$

where $\varepsilon = v_F p - \mu$. See the last (BCS) integral in¹⁹.

B. Cubic terms coefficients calculation

To fix the two coefficients, β_1 and β_2 in Eq.(32) we use only two components. The particular case $j = k = l = 1$ (the coefficient of $\psi_1^{*2}\psi_1$) gives after angle integration

$$\begin{aligned} \beta_1 + \beta_2 &= \frac{2T}{15\pi^2} \sum_n \int_{p=0}^{\infty} \quad (70) \\ &= \frac{p^2 \left(p^4 + 10p^2 (\omega_n^2 - 5\mu^2) - 15 (\mu^2 + \omega_n^2)^2 \right)}{\left(p^4 + 2p^2 (\omega_n^2 - \mu^2) + (\mu^2 + \omega_n^2)^2 \right)^2} \quad (71) \end{aligned}$$

Performing finite integration (the upper bound on momentum, $\mu + T_D$, can be replaced by infinity), one obtains

$$\beta_1 + \beta_2 = \frac{8\mu^2}{15\pi^4} s_3, \quad (72)$$

where the sum is

$$s_3 = \sum_{n=0} \frac{1}{(2n+1)^3} = \frac{7\zeta(3)}{4}. \quad (73)$$

Similarly taking $j = l = 2, k = 1$ (the coefficient of $\psi_2^{*2}\psi_1$) gives after the angle integration

$$\beta_2 = \frac{2T}{15\pi^2} \sum_n \int_{p=0}^{\infty} \quad (74)$$

$$\frac{p^2 \left(7p^4 + 10p^2 (\omega_n^2 + 3\mu^2) + 15 (\mu^2 + \omega_n^2)^2 \right)}{\left(p^4 + 2p^2 (\omega_n^2 - \mu^2) + (\mu^2 + \omega_n^2)^2 \right)^2} \quad (75)$$

$$= -\frac{4\mu^2}{15\pi^4 T^2} s_3. \quad (76)$$

resulting in Eq.(39).

C. Effect of the Pauli interaction

The single particle Hamiltonian in magnetic field was written in Eq.(66). In order to fix the coefficient of the paramagnetic term linear in both the order parameter and Pauli coupling it is enough to expand the linearized Gorkov equations Eq.(35) to the first order in the spin density. Normal Greens functions have the following corrections:

$$\begin{aligned} D_Z &\approx D - \mu_B D (\Sigma \cdot B) D; \\ \tilde{D}_Z &\approx \tilde{D} + \mu_B \tilde{D} (\Sigma^t \cdot B) \tilde{D}. \end{aligned} \quad (77)$$

The Pauli term in Gor'kov equation (after multiplying by T_i^t and taking the trace as usual), Eq.(32), therefore is obtained from expansion of Eq.(35),

$$\sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_i \tilde{D} \Delta^* D \right\} \quad (78)$$

$$\begin{aligned} &= -i\mu_Z \varepsilon_{ijk} \Delta_j^* B_k N(\mu) = \mu_B \mathcal{B}_{ijk}^Z \Delta_j^* B_k, \quad (79) \\ \mathcal{B}_{ijk}^Z &= \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_i \tilde{D} \left(\Sigma_k^t \tilde{D} T_j^* - T_j^* D \Sigma_k \right) D \right\}, \end{aligned}$$

The bubble sum is directly evaluated and vanishes $\mathcal{B}_{ijk}^Z = 0$.

IX. APPENDIX B. CALCULATION OF GRADIENT TERMS

Rotational invariance allows to represent the sum in Eq (46) terms of coefficients u_T and u_L :

$$-N(\mu) (u_T (P^2 \delta_{mj} - P_m P_j) + u_L P_m P_j) \quad (80)$$

$$= P_k P_l \sum_{\omega \mathbf{q}} \text{Tr} \left\{ T_m^t \tilde{D}(\omega, \mathbf{q}) T_j^{t*} D_{kl}''(\omega, \mathbf{q}) \right\}, \quad (81)$$

where

$$\begin{aligned} D_{ij}'' &= \frac{2}{\left(q^2 - (i\omega + \mu)^2 \right)^2} \\ &\quad \{ q_j \alpha_i + q_i \alpha_j + \delta_{ij} (i\omega + \mu + \boldsymbol{\alpha} \cdot \mathbf{q}) + 2q_i q_j D \} \end{aligned} \quad (82)$$

In particular

$$\begin{aligned} N(\mu) u_L &= - \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_z \tilde{D}(\omega, \mathbf{p}) T_z^* D_{zz}''(\omega, \mathbf{p}) \right\} \\ &= \frac{\mu^2}{15\pi^4 T^2 v_F \hbar} s_3 \\ &= \frac{7\zeta(3)}{60\pi^4} \frac{\mu^2}{T^2 v_F \hbar} = \frac{7\zeta(3) (v_F \hbar)^2 N(\mu)}{120\pi^2 T^2} \end{aligned}$$

and

$$N(\mu) u_T = - \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T_x \tilde{D}(\omega, \mathbf{p}) T_x^* D_{xx}''(\omega, \mathbf{p}) \right\} = 32 u_L.$$

X. APPENDIX C. EXACT SOLUTION FOR UPPER CRITICAL MAGNETIC FIELD

In this Appendix the matrix \mathcal{H} defined in Eq.(61) determining the perpendicular upper critical field is diagonalized variationally.

A. Creation and annihilation operators

Using Landau creation and annihilation operators in units of magnetic length $\frac{e^* B}{c} = l^{-2}$ for the state with $k_x = 0$ (independent of x), so that covariant derivatives are

$$D_x = \partial_x + iy = iy = \frac{i}{\sqrt{2}} (a + a^+); \quad (84)$$

$$D_y = \partial_y = \frac{1}{\sqrt{2}} (a - a^+).$$

In terms of these operators the matrix operator \mathcal{H} takes a form:

$$\mathcal{H} = u_T + \frac{u}{2} + \mathcal{V}; \quad (85)$$

$$\mathcal{V}_{11} = 2u_T a^+ a + \frac{u}{2} (a^2 + a^{+2} + 2a^+ a) \quad (86)$$

$$\mathcal{V}_{12} = \mathcal{V}_{21} = \frac{i u}{2} (a^{+2} - a^2) \quad (87)$$

$$\mathcal{V}_{22} = 2u_T a^+ a - \frac{u}{2} (a^2 + a^{+2} - 2a^+ a) \quad (88)$$

The exact lowest eigenvalue is a combination of two lowest Landau levels. Indeed applying the operator \mathcal{V} on

a general vector on the subspace gives

$$\begin{aligned} & \mathcal{V} \begin{pmatrix} \alpha |0\rangle + \beta |2\rangle \\ \gamma |0\rangle + \delta |2\rangle \end{pmatrix} \\ &= \begin{pmatrix} \frac{u}{\sqrt{2}} (i\delta - \beta) |0\rangle \\ - \left(u \left(\frac{\alpha}{\sqrt{2}} + \frac{i\gamma}{\sqrt{2}} + 2\beta \right) + 4u_T\beta \right) |2\rangle \\ + \frac{u}{2} (-\beta - i\delta) |4\rangle \\ + \frac{u}{\sqrt{2}} (i\beta + \delta) |0\rangle \\ + \left(u \left(-\frac{i\alpha}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} - 2\delta \right) - 4u_T\delta \right) |2\rangle \\ + \frac{u}{2} (-i\beta + \delta) |4\rangle \end{pmatrix}. \end{aligned} \quad (89)$$

For $\delta = i\beta$, higher Landau levels decouple and one gets

eigenvalue equations

$$\begin{vmatrix} -v & -\sqrt{2}u & 0 \\ -\frac{u}{\sqrt{2}} & -4u_T - 2u - v & -\frac{iu}{\sqrt{2}} \\ 0 & ui\sqrt{2} & -v \end{vmatrix} = 0,$$

resulting in three eigenvalues of \mathcal{H}

$$h^{(1)} = u_T + u/2, h^{(\pm)} \quad (90)$$

$$= 3u_T + \frac{3}{2}u \pm \sqrt{4u_T^2 + 4u_Tu + 3u^2}. \quad (91)$$

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